Exam Calculus 2

30 March 2016, 14:00-17:00

university of groningen

The exam consists of 6 problems. You have 180 minutes to answer the questions. You can achieve 100 points which includes a bonus of 10 points.

1. [**3+4+4+4 Points.**] Let the map $f : \mathbb{R}^n \to \mathbb{R}$ be defined as

$$\mathbf{x} = (x_1, \ldots, x_n) \mapsto \|\mathbf{x}\| = \sqrt{x_1^2 + \ldots + x_n^2}.$$

   (a) Compute the partial derivatives of f at $(x_1, \ldots, x_n) \neq (0, \ldots, 0)$.

   (b) Show that f is not differentiable at $(x_1, \ldots, x_n) = (0, \ldots, 0)$.

   (c) In which directions $\boldsymbol{v} = (v_1, \ldots, v_n) \in \mathbb{R}^n$, $\|\boldsymbol{v}\| = 1$, do the directional derivatives of f exist at $(x_1, \ldots, x_n) = (0, \ldots, 0)$?

   (d) The Laplacian of a C^2 function $g : \mathbb{R}^n \to \mathbb{R}$ is denoted as $\nabla^2 g$ and defined as

$$\nabla^2 g = \frac{\partial^2 g}{\partial x_1^2} + \ldots + \frac{\partial^2 g}{\partial x_n^2}.$$

   Suppose that $g(x_1, \ldots, x_n) = h(\|\mathbf{x}\|)$ for some C^2 function $h : \mathbb{R} \to \mathbb{R}$. Show that for $(x_1, \ldots, x_n) \neq (0, \ldots, 0)$, the Laplacian of such a function g is given by

$$\nabla^2 g(x_1, \ldots, x_n) = \frac{n-1}{\|\mathbf{x}\|} h'(\|\mathbf{x}\|) + h''(\|\mathbf{x}\|).$$

2. [**7+3+5 Points.**] Consider the curve parametrized by $\mathbf{r} : [0, 2\pi] \to \mathbb{R}^3$ with

$$\mathbf{r}(t) = a \cos t \, \mathbf{i} + a \sin t \, \mathbf{j} + bt \, \mathbf{k},$$

   where a and b are positive constants.

   (a) Determine the parametrization by arc length.

   (b) For each point on the curve, determine a unit tangent vector.

   (c) At each point on the curve, determine the curvature of the curve.

3. [**5+10 Points.**] Consider the ellipsoid $x^2 + 2y^2 + 3z^2 = 6$.

   (a) Compute the tangent plane at the point $(x, y, z) = (1, -1, -1)$.

   (b) Use the Method of Lagrange Multipliers to find the points on the ellipsoid which have minimal and maximal distance to the origin.

– please turn over —

4. **[5+7+3 Points.]**

   Let the vector field $\mathbf{F} : \mathbb{R}^3 \to \mathbb{R}^3$ be defined as $\mathbf{F}(x, y, z) = (2xy + z^3)\,\mathbf{i} + x^2\,\mathbf{j} + 3xz^2\,\mathbf{k}$.

   (a) Show that $\mathbf{F}$ is conservative.

   (b) Determine a potential function for $\mathbf{F}$.

   (c) Compute the line integral of $\mathbf{F}$ along the straight line segment from the point $(1, -2, 1)$ to the point $(3, 1, 4)$.

5. **[15 Points.]**

   Verify Stokes' Theorem for the surface S defined as $x^2 + y^2 + 5z = 1$, $z \geq 0$, oriented by the upward normal and the vector field $\mathbf{F}(x, y, z) = xz\,\mathbf{i} + yz\,\mathbf{j} + (x^2 + y^2)\,\mathbf{k}$.

6. **[7+8 Points.]**

   For $(x, y, z) \neq (0, 0, 0)$, let the vector field $\mathbf{F} : \mathbb{R}^3 \to \mathbb{R}^3$ be defined as

$$\mathbf{F}(x, y, z) = \frac{1}{(x^2 + y^2 + z^2)^{3/2}}(x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}).$$

   (a) Let S_a be the sphere of radius $a > 0$ centered at the origin in $\mathbb{R}^3$. Compute the flux of $\mathbf{F}$ though S_a where S_a is oriented by the outward pointing normal.

   (b) Use Gauss' Divergence Theorem to show that the flux of $\mathbf{F}$ through

      i. any closed surface S which encloses a solid region not containing the origin is zero and

      ii. any closed surface S which encloses a solid region that contains the origin is 4π (for this case, make use of part (a)).

Solutions

1. (a) For $k = 1, \ldots, n$,

$$\frac{\partial f}{\partial x_k}(\mathbf{x}) = \frac{\partial}{\partial x_k}\sqrt{x_1^2 + \ldots + x_n^2} = 2x_k \frac{1}{2}(x_1^2 + \ldots + x_n^2)^{-1/2} = \frac{x_k}{\|\mathbf{x}\|}.$$

(b) f is not differentiable at $\mathbf{x} = 0$ because the partial derivatives of f do not exist at $\mathbf{x} = 0$. Consider, e.g., the difference quotient for the derivative with respect to x_1:

$$\frac{f(t, 0, \ldots, 0) - f(0, \ldots, 0)}{t} = \frac{\sqrt{t^2 + 0^2 + \ldots + 0^2}}{t} = \frac{|t|}{t}$$

which is 1 for $t > 0$ and -1 for $t < 0$. So the difference quotient has no limit for $t \to 0$.

(c) The directional derivative of f does not exist in any direction $\boldsymbol{v}$. This is because the difference quotient

$$\frac{f(t\boldsymbol{v}) - f(0)}{t} = \frac{\sqrt{(tv_1)^2 + \ldots (tv_n)^2} - 0}{t} = \frac{|t|\sqrt{(v_1)^2 + \ldots (v_n)^2}}{t} = \frac{|t|\|\boldsymbol{v}\|}{t}$$

is 1 or -1 depending on whether t is positive or negative. So the limit $t \to 0$ does not exist.

(d) By the chain rule we have for $k = 1, \ldots, n$,

$$\frac{\partial}{\partial x_k} h(\|\mathbf{x}\|) = \frac{\partial}{\partial x_k} h(f(\mathbf{x})) = h'(f(\mathbf{x}))\frac{\partial f(\mathbf{x})}{\partial x_k} = h'(f(\mathbf{x}))\frac{x_k)}{\|\mathbf{x}\|}.$$

Again differentiating with respect to x_k gives

$$\begin{aligned}
\frac{\partial^2}{\partial x_k^2} h(\|\mathbf{x}\|) &= \frac{\partial}{\partial x_k} h'(f(\mathbf{x}))\frac{x_k)}{\|\mathbf{x}\|} \\
&= h''(f(\mathbf{x}))\left(\frac{x_k)}{\|\mathbf{x}\|}\right)^2 + h'(f(\mathbf{x}))\frac{\partial}{\partial x_k}\frac{x_k}{\|\mathbf{x}\|} \\
&= h''(f(\mathbf{x}))\left(\frac{x_k)}{\|\mathbf{x}\|}\right)^2 + h'(f(\mathbf{x}))\frac{\|\mathbf{x}\| - x_k \frac{x_k}{\|\mathbf{x}\|}}{\|\mathbf{x}\|^2} \\
&= h''(\|\mathbf{x}\|)\frac{x_k^2}{\|\mathbf{x}\|^2} + h'(\|\mathbf{x}\|)\frac{\|\mathbf{x}\| - \frac{x_k^2}{\|\mathbf{x}\|}}{\|\mathbf{x}\|^2},
\end{aligned}$$

where in the first equality we again used the chain rule and the product rule, and in the second equality we used the quotient rule. Summing over k gives the desired equality.

2. (a) The tangent vector

$$\mathbf{r}'(t) = -a\sin t\mathbf{i} + a\cos t\,\mathbf{j} + b\,\mathbf{k}$$

has length

$$\|\mathbf{r}'(t)\| = \sqrt{a^2\sin^2 t + a^2\cos^2 t + b^2} = \sqrt{a^2 + b^2}\,.$$

For $t \in [0, 2\pi]$, the arc length is hence

$$s(t) = \int_0^t \|\mathbf{r}'(\tau)\|\,\mathrm{d}\tau = t\sqrt{a^2 + b^2}.$$

Inverting for t gives

$$t(s) = \frac{s}{\sqrt{a^2+b^2}}.$$

The parametrization by arc length is hence given by

$$\tilde{\mathbf{r}}(s) = \mathbf{r}(t(s)) = a\cos\frac{s}{\sqrt{a^2+b^2}}\,\mathbf{i} + a\sin\frac{s}{\sqrt{a^2+b^2}}\,\mathbf{j} + b\frac{s}{\sqrt{a^2+b^2}}\,\mathbf{k}\,.$$

(b) The unit tangent vector is given by

$$\mathbf{T} \;=\; \frac{\mathrm{d}\tilde{\mathbf{r}}(s)}{\mathrm{d}s} = -\frac{a}{\sqrt{a^2+b^2}}\sin\frac{s}{\sqrt{a^2+b^2}}\,\mathbf{i} + \frac{a}{\sqrt{a^2+b^2}}\cos\frac{s}{\sqrt{a^2+b^2}}\,\mathbf{j} + b\frac{1}{\sqrt{a^2+b^2}}\,\mathbf{k}$$

which agrees with

$$\mathbf{T} = \frac{1}{\|\mathbf{r}'(t)\|}\mathbf{r}'(t)$$

for $t = s/\sqrt{a^2+b^2}$.

(c) The curvature is given by

$$\begin{aligned}\kappa \;&=\; \left\|\frac{\mathrm{d}\mathbf{T}}{\mathrm{d}s}\right\| = \left\|\frac{a}{a^2+b^2}\cos\frac{s}{\sqrt{a^2+b^2}}\,\mathbf{i} + \frac{a}{a^2+b^2}\sin\frac{s}{\sqrt{a^2+b^2}}\,\mathbf{j}\right\| \\ &=\; \frac{a}{a^2+b^2}.\end{aligned}$$

which agrees with

$$\left\|\frac{\mathrm{d}\mathbf{T}}{\mathrm{d}t}\right\|\frac{1}{\left\|\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}\right\|}$$

for $t = s/\sqrt{a^2+b^2}$.

3. (a) Let $g(x,y,z) = x^2+2y^2+3z^2$. Then the ellipsoid is the level set of g with value 6. The tangent plane of the ellipsoid at the point $(1,-1,-1)$ is perpendicular to the gradient vector of g at $(1,-1,-1)$. The gradient of g is

$$\nabla g(x,y,z) = (2x, 4y, 6z)$$

giving

$$\nabla g(x,y,z) = (2,-4,-6).$$

The tangent plane is hence given by

$$(2,-4,-6)\cdot(x-1, y+1, z+1) = 0$$

which gives

$$2x - 4y - 6z = 12.$$

(b) Let $f(x,y,z) = x^2+y^2+z^2$ be the squared distance to the origin. We need to find the extrema of f under the constraint $g(x,y,z) = 6$ with g defined as in part (a). At the extremal points there is according to the theorem on Lagrange multipliers a $\lambda \in \mathbb{R}$ such that $\lambda\nabla f(x,y,z) = \nabla g(x,y,z)$. Together with the constraint $g(x,y,z) = 6$ this gives the following four scalar equations:

$$\begin{aligned}\lambda f_x(x,y,z) \;&=\; g_x(x,y,z),\\ \lambda f_y(x,y,z) \;&=\; g_y(x,y,z),\\ \lambda f_z(x,y,z) \;&=\; g_z(x,y,z),\\ g(x,y,z) \;&=\; 6\end{aligned}$$

i.e.

$$\begin{aligned} 2\lambda x &= 2x, \\ 2\lambda y &= 4y, \\ 2\lambda z &= 6z, \\ x^2 + 2y^2 + 3z^2 &= 6. \end{aligned}$$

This is equivalent to

$$\begin{array}{rcl} x = 0 & \text{or} & \lambda = 1, \\ y = 0 & \text{or} & \lambda = 2, \\ z = 0 & \text{or} & \lambda = 3, \\ x^2 + 2y^2 + 3z^2 & = & 6. \end{array}$$

As λ needs to be unique the only possible solutions are

i. $x = y = 0,\ \lambda = 3,\ z = \pm\sqrt{2}$,

ii. $x = z = 0,\ \lambda = 2,\ y = \pm\sqrt{3}$,

iii. $y = z = 0,\ \lambda = 1,\ x = \pm\sqrt{6}$,

We have $f(0, 0, \pm\sqrt{2}) = 2,\ f(0, \pm\sqrt{3}, 0) = 3$ and $f(\pm\sqrt{6}, 0, 0) = 6$. So the distance is minimal at $(x, y, z) = (0, 0, \pm\sqrt{2})$ and maximal at $(x, y, z) = (\pm\sqrt{6}, 0, 0)$.

4. (a) To show that $\mathbf{F}$ is conservative we show that the curl of $\mathbf{F}$ is vanishing:

$$\begin{aligned} \text{curl}\,\mathbf{F}(x, y, z) &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial_x & \partial_y & \partial_z \\ 2xy + z^3 & x^2 & 3xz^2 \end{vmatrix} \\ &= \left(\partial_y 3xz^2 - \partial_z x^2\right)\mathbf{i} - \left(\partial_x 3xz^2 - \partial_z(2xy + z^3)\right)\mathbf{j} + \left(\partial_x x^2 - \partial_y(2xy + z^3)\right)\mathbf{k} \\ &= 0\,\mathbf{i} - (3z^2 - 3z^2)\,\mathbf{j} + (2x - 2x)\,\mathbf{k} \\ &= 0 \end{aligned}$$

(b) Let f denote the potential function. Then f satisfies the equations

$$\begin{aligned} f_x &= 2xy + z^3, & (1) \\ f_y &= x^2, & (2) \\ f_z &= 3xz^2. & (3) \end{aligned}$$

Integrating the first equation with respect to x gives

$$f(x, y, z) = x^2y + xz^3 + h(y, z),$$

where $h(y, z)$ is a integration constant which can dependent on y and z. Differentiating with respect to y and using Equation (2) yields

$$x^2 + h_y(y, z) = x^2,$$

i.e., $h_y(y, z) = 0$. So h does not dependent on y and is hence of the form $h(y, z) = g(z)$ for some function g. So $f(x, y, z) = x^2y + xz^3 + g(z)$. Differentiating with respect to z and using Equation (3) yields

$$3xz^2 + g'(z) = 3xz^2$$

which gives $g'(z) = 0$, i.e. g is constant. So the potential function is

$$f(x, y, z) = x^2y + xz^3 + c$$

with $c \in \mathbb{R}$.

(c) According to the Fundamental Theorem for Line Integrals the line integral is given by $f(3,1,4) - f(1,-2,1)$, where f is the potential function computed in part (b). As $f(3,1,4) = 201$ and $f(1,-2,1) = -1$ the line integral is 202.

5. To Verify Stokes' Theorem we need to show that

$$\iint_S \nabla \times \mathbf{F} \cdot \mathrm{d}\mathbf{S} = \int_{\partial S} \mathbf{F} \cdot \mathrm{d}\mathbf{s}. \tag{4}$$

The surface S is the part of a downward open paraboloid having $z \geq 0$. Its boundary ∂S is given the circle $x^2 + y^2 = 1$, $z = 0$. This circle has the parametrization

$$\mathbf{r}(t) = \cos t\,\mathbf{i} + \sin t\,\mathbf{j} + 0\,\mathbf{k}$$

with $t \in [0, 2\pi]$. As S is oriented by the upward normal the parametrization $\mathbf{r}$ gives an orientation of ∂S which is consistent with that of S. The right hand side of Equation (4) is

$$\begin{aligned}\int_{\partial S} \mathbf{F} \cdot \mathrm{d}\mathbf{s} &= \int_0^{2\pi} \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\,\mathrm{d}t \\ &= \int_0^{2\pi} (0\,\mathbf{i} + 0\,\mathbf{j} + (\cos^2 t + \sin^2 t)\,\mathbf{k}) \cdot (-\sin t\,\mathbf{i} + \cos t\,\mathbf{j} + 0\,\mathbf{k})\,\mathrm{d}t = 0.\end{aligned}$$

In order to compute the left hand side of Equation (4) we use the parametrization $X(s,t) = s\,\mathbf{i} + t\,\mathbf{j} + \frac{1}{5}(1 - s^2 - t^2)\,\mathbf{k}$ with $(s,t) \in D = \{(s,t) \in \mathbb{R}^2 \mid s^2 + t^2 \leq 1\}$. The corresponding tangent vectors are

$$\frac{\partial X}{\partial s} = \mathbf{i} - \frac{2}{5}s\,\mathbf{k}$$

and

$$\frac{\partial X}{\partial t} = \mathbf{j} - \frac{2}{5}t\,\mathbf{k}.$$

This gives the normal vector

$$\mathbf{N}(s,t) = \frac{\partial X}{\partial s} \times \frac{\partial X}{\partial t} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & -\frac{2}{5}s \\ 0 & 1 & -\frac{2}{5}t \end{vmatrix} = \frac{2}{5}s\,\mathbf{i} + \frac{2}{5}t\,\mathbf{j} + 1\,\mathbf{k}$$

which is consistent with orientation of S by the upward normal. The curl of $\mathbf{F}$ is

$$\nabla \times \mathbf{F}(x,y,z) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial_x & \partial_y & \partial_z \\ xz & yz & x^2 + y^2 \end{vmatrix} = y\,\mathbf{i} - x\,\mathbf{j}.$$

As $\nabla \times \mathbf{F}(x,y,z) \cdot \mathbf{N} = 0$ the right hand side of Equation (4) vanishes.

6. (a) An outward pointing unit normal vector of the sphere S_a at the point (x,y,z) is given by

$$\mathbf{n}(x,y,z) = \frac{1}{a}(x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}).$$

The flux integral is

$$\begin{aligned}\iint_{S_a} \mathbf{F}\cdot\mathbf{n}\,\mathrm{d}S &= \frac{1}{a}\iint_{S_a} \frac{1}{(x^2+y^2+z^2)^{3/2}}(x\,\mathbf{i}+y\,\mathbf{j}+z\,\mathbf{k})\cdot(x\,\mathbf{i}+y\,\mathbf{j}+z\,\mathbf{k})\,\mathrm{d}S\\ &= \frac{1}{a}\iint_{S_a}\frac{1}{a^3}a^2\,\mathrm{d}S\\ &= 4\pi,\end{aligned}$$

where we used that the surface area of a sphere of radius a is $4\pi a^2$.

(b) At $(x, y, z) \neq 0$ the divergence of $\mathbf{F}$ is

$$\begin{aligned}\nabla\cdot\mathbf{F}(x,y,z) &= \partial_x\frac{x}{(x^2+y^2+z^2)^{3/2}} + \partial_y\frac{y}{(x^2+y^2+z^2)^{3/2}} + \partial_z\frac{z}{(x^2+y^2+z^2)^{3/2}}\\ &= \frac{(x^2+y^2+z^2)^{3/2} - 3x^2(x^2+y^2+z^2)^{1/2}}{(x^2+y^2+z^2)^3} +\\ &\quad \frac{(x^2+y^2+z^2)^{3/2} - 3y^2(x^2+y^2+z^2)^{1/2}}{(x^2+y^2+z^2)^3} +\\ &\quad \frac{(x^2+y^2+z^2)^{3/2} - 3z^2(x^2+y^2+z^2)^{1/2}}{(x^2+y^2+z^2)^3}\\ &= 0.\end{aligned}$$

Let the solid region enclosed by S be denoted as D. If the origin is not contained in D then we get for the flux through S from Gauss' Theorem

$$\iint_S \mathbf{F}\cdot\mathrm{d}\mathbf{S} = \iiint_D \nabla\cdot\mathbf{F}\,\mathrm{d}V = 0.$$

If D contains the origin then define a little ball of radius a around the origin with a sufficiently small such that the ball is contained in D. Now consider the modified region D' given by D minus the ball. The boundary of D' is S minus the sphere S_a. Applying Gauss' Theorem to the D' gives

$$0 = \iiint_{D'} \nabla\cdot\mathbf{F}\,\mathrm{d}V = \iint_S \mathbf{F}\cdot\mathrm{d}\mathbf{S} - \iint_{S_a}\mathbf{F}\cdot\mathrm{d}\mathbf{S},$$

where the minus in front of the second term comes from the fact that in order to apply Gauss' Theorem we need to orient the sphere S_a opposite to the orientation used in part (a). Using the result for the flux through S_a in part (a) we find

$$\iint_S \mathbf{F}\cdot\mathrm{d}\mathbf{S} = 4\pi.$$